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A Model of Changing Net Maternity Rates Leading to Stability

The Problem

RECENTLY Frauenthal and Swick (1983) have made important contributions in the theory of renewal process by incorporating in the Lotka model Easterlin's (1968) idea that fertility is affected by the size of the cohort in which an individual is born. They begin by writing the birth function as

$$B(t) = \int_0^\infty B(t-s)\, l(s, t) m(s, t) ds \tag{1}$$

for large t where $B(t)$ is the number of female births at time t ($t > \beta$, the upper limit of the reproductive interval), $l(s, t)$ is the probability of surviving from birth to age s at time t and $m(s, t)ds$ is the probability that a woman born at time $t - s$ will bear a female child when she is in the age interval s, $s + ds$. Next, they assume $l(s, t)$ as independent of time or $l(s, t) = l(s)$ but allow $m(s, t)$ to vary inversely with the size of $B(t - s)$ following Easterlin's idea. They then write

$$m(s, t) = m(s)\, M\,\{B(t-s)\} \tag{2}$$

and (1) as

$$B(t) = \int_0^\infty B(t-s)\, \phi(s)\, M\,\{B(t-s)\} ds \tag{3}$$

where $\phi(s) = l(s)\, m(s)$ is the net maternity function. For operational convenience they have chosen $m(s)$ in a manner such that

$$R_0 = \int_0^\infty \phi(s) ds = 1 \tag{4}$$

It may be noted that $M\{B(t-s)\}$ being dependent on $t-s$ is such that it remains the same for a given cohort for all ages. This is so because at time $t+1$ this cohort will be $s+1$ years old and therefore,

$$M\{B(t+1-s+1)\} = M\{B(t-s)\} \tag{5}$$

due to which this cohort's net reproduction rate

$$R_0\{B(t-s)\} = M\{B(t-s)\} \tag{6}$$

This means that the age specific fertility rates of two cohorts of women will maintain a constant ratio at all ages. Frauenthal and Swick then note that $M'\{B(t-s)\} \leqslant 0$ and that at the point of stationarity where $B = E$ (which is yet arbitrary),

$$M(E) = 1 \tag{7}$$

They then assume that

$$M'(E) = -\gamma/E, \gamma > 0 \tag{8}$$

which enable them to write the following equation

$$B(t) - E = (1-\gamma)\int_0^\infty \{B(t-s) - E\}\, \phi(s)\, ds \tag{9}$$

by writing Taylor series expansion of $B(t-s)\ M\{B(t-s)\}$ at the point E and ignoring higher order terms. They then proceed to estimate the parameters from (9) which has now assumed the form of the well known linear integral equation.

The purpose of this paper is to show that there exists a simple functional form of $M\{B(t-s)\}$ that would make (9) an identity rather than an approximation. It will also be seen from that form that the parameter γ can be given a meaningful interpretation that is consistent with the changes in the reproductive behaviour required by the model. The same will be attempted on the model without forcing condition (4) and a slightly different approach will be taken to arrive at a solution comparable to (9).

A Simple Functional Form of $M\{B(t-s)\}$

For a stationary model, the limiting size E can be approached subject to an inverse relationship between $M\{B(t-s)\}$ and $B(t-s)$, such as

$$M\{B(t-s)\} = \frac{\gamma E}{B(t-s)} + (1-\gamma) \tag{10}$$

It can be easily seen that $M\{B(t-s)\}$ so defined meets the conditions outlined in (7) and (8) and its substitution in (3) and subsequent simplification results in

(9). While it is true that from a mathematical point of view the integral equation (9) can be studied for all positive values of y, the dynamics of reproductive behaviour cannot be meaningfully interpreted unless $0 \leq \gamma \leq 1$. This is so because the model is based on the assumption that fertility has to vary inversely with the size of the birth cohort $B(t - s)$ and from that point of view (10) can be understood in the following sense. At any given point of time a fixed proportion of the birth cohort, say γ respond to the social changes by adjusting their fertility in the prescribed manner while the remainder, namely, $1 - \gamma$ remain unaffected by it and maintain stationarity. Such an interpretation of (10) is not possible if $\gamma > 1$.

Observe that this particular approach is suitable only for a stationary mode], and in order to demonstrate its fit with the U. S. examples, Frauenthal and Swick replaced the time series of U. S. birth statistics $B(t - s)$ which is not stationary by a stationary series $B^*(t - s)$ using suitable inflation and deflation factors that were determined by the average rate of growth.

They then proceeded to determine an empirical relationship between $M\{B(t - s)\}$ or $R_0\{B(t - s)\}$ and the adjusted $B^*(t - s)$ and came up with a suggestion of a regression model of $R_0\{B(t - s)\}$ on $(B^*(t - s))^8$ based on the technique of exploratory regression. However, the major drawback of this technique is that the form of the model is dependent too much on the number of data points, such that the addition or subtraction of a number of observations will, in all likelihood, alter the form greatly unless, of course, the fit is perfect. Moreover, such a model in which the cohort net reproduction rate has to be regressed on the eighth power of the adjusted size of the cohort is difficult to justify on theoretical grounds and especially so, when (9) is fully determined by (10). It is possible that the authors were not aware of the true nature of the relationship and therefore, in their search for an empirical formula, they settled on the aforementioned regression equation that was based on a large value of the correlation coefficient.

In the following, we shall note the modification required in (10) that will make it applicable to stable models in general and will satisfy the conditions for the stationary example as a paiticular case. We shall then have a general model that can account for the birth trajectory of any population in which the birth rates fluctuate according to Easterlin's hypothesis. In order to develop such a^model, we shall, as before, propose that a proportion $1 - \gamma$ of the birth cohort $B(t - s)$ will be subjected to the constant net maternity rates while the remainder will adjust their rates relative to their expected sizes. If the underlying intrinsic rate of growth is ρ, the expected size of this birth cohort may be regarded as $B_0 e^{\rho(t-s)}$ where B_0 is to be estimated, in which case a generalized version of (10) may be written as

$$M\{B(t - s)\} = \frac{\gamma B_0 e^{\rho(t-s)}}{B(t - s)} + (1 - \gamma) \tag{11}$$

The nature of $M\{B(t-s)\}$ for the limiting values of γ may be noted at this point. When $\gamma = 0$, $M\{B(t-s)\} = 1$ which is equivalent to the standard *assumptions leading to a stable population, namely, that the net maternity* function $\phi(s)$ is independent of time (see equation 3). When $\gamma = 1$, substitution of (11) in (3) shows $B(t)$ as an exponential function of t for all t.

The difference between the two cases is that in the former, stability of the birth trajectory is achieved as $t \rightarrow \infty$, whereas in the latter, it is achieved at the very moment the model becomes operative. It will be interesting to examine the nature of the solutions for all other values of γ, which has been shown next.

Solution of the Integral Equation

Substitution of (11) in (3) gives

$$B(t) = \gamma B_0 e^{\rho t} \int_0^{\infty} e^{-\rho s}\, \phi(s)ds + (1-\gamma)\int_0^{\infty} B(t-s)\, \phi(s)ds \qquad (12)$$

Since the continuous operation of $\phi(s)$ produces ρ, such that

$$\int_0^{\infty} e^{-\rho s}\, \phi(s)ds = 1 \qquad (13)$$

(12) reduces to

$$B(t) = \gamma B_0\, e^{\rho t} + (1-\gamma)\int_0^{\infty} B(t-s)\, \phi(s)ds \qquad (14)$$

In order to solve (14), we first rewrite it as

$$B(t)e^{-\rho t} = \gamma B_0 + (1-\gamma)\int_0^{\infty} B(t-s)e^{-\rho(t-s)}e^{-\rho s}\, \phi(s)ds \qquad (15)$$

It may be seen from (15) that the birth function as well as the net maternity rates have been adjusted for a stable rate of growth as Frauenthal and Swick have proposed. It may be noted that as it is, (15) can neither be inferred from (9), nor can it be derived from (9) in a straightforward manner. The converse is, however, possible as it should be.

Now for the solution of (15) we begin by writing

$$F(t) = B(t)e^{-\rho t} \qquad (16)$$

so that it can be expressed as

$$F(t) = \gamma B_0 + (1-\gamma)\int_0^{\infty} F1(t-s)e^{-\rho s}\phi(s)ds \qquad (17)$$

An equation similar to (17) was earlier solved by Mitra (1983) by assuming continuity and differentiability of $F(t)$. Thus

$$F'(t) = (1 - \gamma)\int_0^\infty F'(t - s)e^{-\rho s} \phi(s)ds \tag{18}$$

When $0 < \gamma < 1$

$$0 < \int_0^\infty (1 - \gamma)e^{-\rho s} \phi(s)ds < 1 \tag{19}$$

Therefore,

$$\lim_{t \to \infty} F'(t) = 0 \tag{20}$$

and so

$$\lim_{t \to \infty} F(t) = \lim_{t \to \infty} B(t)e^{-\rho t} = C \tag{21}$$

where C is a constant, which due to the initial condition should be equal to B_0 so that

$$\lim_{t \to \infty} B(t) = B_0 e^{\rho t} \tag{22}$$

It may be noted from (14) that the same solution is obtained when $\gamma = 0$ or when $\gamma = 1$.

In order to estimate γ, we begin by writing the net reproduction rate of the birth cohort $B(t - s)$, as

$$R_0\{B(t - s)\} = \int_0^\infty M\{B(t - s)\} \phi(a)da \tag{23}$$

$$= M\{B(t - s)\} R_0(c)$$

since for a given birth cohort the factor $M\{B(t - s)\}$ remains the same at all ages (see equation 5) and $R_0(c)$ is the underlying net reproduction rate. Therefore,

$$M\{B(t - s)\} = \frac{R_0\{B(t - s)\}}{R_0(c)} \tag{24}$$

so that we can write (11) as

$$R_0\{B(t - s)\} = \frac{\gamma R_0(c) B_0 e^{\rho(t-s)}}{B(t - s)} + (1 - \gamma) R_0(c) \tag{25}$$

Equation (25) describes a linear relationship between $R_0\{B(t-s)\}$ and $B_0e^{\rho(t-s)}/B(t-s)$ in which the intercept

$$K = (1-\gamma)\, R_0(c) \tag{26}$$

and the slope

$$L = \gamma\, R_0(c) \tag{27}$$

Thus

$$\gamma = \frac{L}{K+L} \tag{28}$$

and

$$R_0(c) = K + L \tag{29}$$

Estimation of Parameters

The readers may recall that the development of the present model is based on the assumption that the population's inherent stability is distorted by irregularities in the birth trajectory. This is reflected in differentials in the age-specific birth rates of the different birth cohorts such that these rates are adjusted by a factor $M\{B(t-s)\}$ as described in (11). If these assumptions are met, then the principal parameters γ and $R_0(c)$ can be solved from (28) and (29). However, prior to that, B_0 and ρ must be known or estimated so that we can regress $R_0\{B(t-s)\}$ on $B_0e^{\rho(t-s)}/B(t-s)$ to estimate K and L from (26) and (27) which in turn will produce estimates of γ and $R_0(c)$.

The logical procedure for estimating B_0 and ρ that suggests itself is to regress $\ln\{B(t-s)\}$ on $t-s$. When this is attempted on the data provided by Frauenthal and Swick in their Table 1, $B_0 = 2.60$ and $\rho = .0052$ are obtained. Note that these values are at variance with those obtained by Frauenthal and Swick since they used the birth statistics for the years 1910-75 instead of the years 1910-54 provided in their Table 1, on which our computations are based. In any event, the substitution of these values in developing the regression of $R_0\{B(t-s)\}$ on $B_0e^{\rho(t-s)}/B(t-s)$ show a correlation coefficient of .85 which is quite encouraging. Subsequent simplifications produce $R_0(c) = 1.16$ which is consistent with the value of ρ. However, the other parameter γ turned out to be greater than 1 ($\gamma = 1.17$) and as such shows that such a model is somewhat less than appropriate for the U. S. data. This is apparent when one examines the fluctuations in the time series of the births and the cohort reproduction rates. Different segments of the birth series produce contrasting values of the rate of growth. This demonstrates that the model based on (9) or (10) does not quite fit the U. S. population trajectory even though Frauenthal *et al.* found it otherwise because they were not concerned with values of ë outside the interval (0, 1) which is somewhat unfortunate.

Concluding Remarks

Even though the model does not fit the U. S. data as well as one would like, the fact that the stability of the birth trajectory may also result from certain types of fluctuating net maternity rates is by itself a significant discovery. The model presented in the paper is particularly interesting since the proposed pattern of the fluctuation in the net maternity rates can be given a meaningful interpretation.

The readers may note that the parameter ë which provides a measures of the proportion of a given birth cohort that adjusts its reproductive behavior in relation to its size does not affect the limiting nature of the birth trajectory. The stable rate of growth p remains unaffected by the value of ë as may be seen from (22). However, the value of ë seems to determine the speed of convergence of the birth trajectory to its stable form. This is apparent from (14) which for ë = 0 reduces to the well known integral equation of the stable population based on the condition that the net maternity rates remain invariant over time. At the other extreme, when ë = 1, the trajectory of births becomes stable from the very moment the model becomes operative. For a given series of $B(t-s)$ values, the rapidity of the convergence to its stable pattern will therefore be greater for larger values of ë and will be less otherwise. The present model can, therefore, be regarded as more general in the sense that it includes the traditional Lotka model as a particular case.

We would like to make another interesting observation concerning the parameter B_0 in this context. That is, B_0 need not be determined by the trend of the known values of some $B(t — s)$ or by any other method since the model can freely accommodate any value of B_0. As long as the underlying net maternity function is given or estimated or assumed, so that we can calculate p, the function $M\{B(t — s)\}$, which determines the fertility of the cohort $B(t — s)$, *can* be obtained from (11) for any value of B_0 and for any value of ë. Alternatively, given $R_Q (e)$ and p and assuming that the model is already in operation, one can estimate B_o, and ë from (25) by coverting it to a linear regression of $R_t\{B(t — s)\}IR_0(c)$ on $e^{f(t-s)}B(t — s)$ for which the intercept and the slope will be given by 1 — y and Y B_0 respectively.

Thus, there are two findings which are interesting and in a sense, somewhat surprising. First, the magnitude of the proportion y of the birth cohort that adjusts its reproductive behavior does not in any way affect the ultimate rate of growth. Second, regardless of the value of ë, the segment of the cohort that adjusts its fertility completely determines the level of the birth trajectory by choosing the value of B_0 at its pleasure. That is to say, if a constant proportion of all birth cohorts adjust their fertility rates in a prescribed manner, the size of the population at any moderately distant point in time can be held at a predetermined value. It seems that such a finding has considerable implications in the areas of population policy and population planning.

References

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